Tentamen Complexe Analyse 06/11/07

1. $0 = z^6 + z^4 + z^2 + 1 = z^4(z^2+1) + z^2 + 1 = (z^4+1)(z^2+1) \Rightarrow z = e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{2}}; \quad z = e^{i\frac{\pi}{4} + k i \frac{\pi}{2}}$

Anders: $0 = z^6 + z^4 + z^2 + 1 = \dfrac{1-z^8}{1-z^2} \Rightarrow z = e^{\frac{k\pi}{4} i}$ m.u.v. $k=0$ en $k=4$

2. • $\{z \in \mathbb{C} : |z| \leqq 1\}$ compact (begrensd en gesloten) $f(z)$ analytisch, dus continu

$\Rightarrow |f(z)|$ continu dwz. er bestaat z_0 met $|z_0| \leqq 1$ met $|f(z_0)| = \max\limits_{|z|\leqq 1} |f(z)|$

Maximum modulus principe $\Rightarrow |z_0| = 1$

• Bepaling van z_0: $|f(z)| = |z^4 e^{-z^2}| = |e^{-z^2}| = e^{-x^2+y^2}$ als $x^2+y^2=1$

$\Rightarrow z_0 = \pm i$ en $|f(z_0)| = e$

3. $\displaystyle\int_0^{2\pi} \sin^4\theta \, d\theta \overset{z=e^{i\theta}}{=\!=\!=} \int_{|z|=1} \left(z - \frac{1}{z}\right)^4 \cdot \frac{1}{(2i)^4} \frac{dz}{iz} = \frac{1}{16i} \int_{|z|=1} (z^2-1)^4 \frac{dz}{z^5} \quad (\dagger)$

$(z^2-1)^2 = z^4 - 2z^2 + 1 \qquad (z^2-1)^4 = (z^4-2z^2+1)(z^4-2z^2+1) = z^8 - 4z^6 + 6z^4 - 4z^2 + 1$

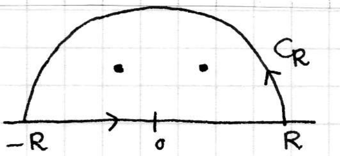
$\Rightarrow \displaystyle\frac{1}{16i} \int_{|z|=1} (z^2-1)^4 \frac{dz}{z^5} = \frac{1}{16i} \cdot 6 \int_{|z|=1} \frac{dz}{z} = 3\frac{\pi}{4}$

Rechtstreeks: $(\dagger)$ $z=0$ is 5^e orde pool: $\displaystyle\frac{1}{16i} 2\pi i \cdot \frac{1}{4!} \left(\frac{d}{dz}\right)^4 (z^2-1)^4 \Big|_{z=0} = \frac{2\pi i}{16 i} \frac{1}{4!} 6 \cdot 4! = \frac{3\pi}{4}$

4. • $\displaystyle G(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{t^4+1} dt = \int_{-\infty}^{\infty} \frac{\cos\omega t - i \sin\omega t}{t^4+1} dt = \int_{-\infty}^{\infty} \frac{\cos\omega t}{t^4+1} dt$ reëel en even

Anders: $\displaystyle G(-\omega) = \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{t^4+1} dt \overset{t=-s}{=\!=\!=} \int_{-\infty}^{\infty} \frac{e^{-i\omega s}}{s^4+1} ds = G(\omega) \Rightarrow \overline{G(\omega)} = G(-\omega) = G(\omega)$ reëel even

• $\omega \leqq 0$



$\displaystyle\int_{-R}^{R} \frac{e^{-i\omega t}}{t^4+1} dt + \int_{C_R} \frac{e^{-i\omega z}}{z^4+1} dz = 2\pi i \left[\mathrm{Res}_{e^{i\frac{\pi}{4}}} + \mathrm{Res}_{e^{3i\frac{\pi}{4}}} \right]$

$\displaystyle = 2\pi i \left[\frac{e^{-i\omega z}}{4z^3} \Big|_{z=e^{i\frac{\pi}{4}}} + \frac{e^{-i\omega z}}{4z^3} \Big|_{z=e^{\frac{3\pi i}{4}}} \right]$

Op C_R: $|e^{-\omega z i}| = e^{\omega y} \leqq 1$ voor $y \geqq 0$, $\omega \leqq 0$ $\displaystyle\Rightarrow \left| \int_{C_R} \frac{e^{-i\omega z}}{z^4+1} dz \right| \leqq \frac{\pi R}{R^4-1} \underset{R\to\infty}{\to} 0$

$\displaystyle\Rightarrow \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{t^4+1} dt = 2\pi i \cdot \frac{1}{4} \left[- e^{-i\omega z} z \Big|_{z=e^{i\pi/4}} - e^{-i\omega z} z \Big|_{z=e^{3\pi i/4}} \right]$

$\displaystyle = -2\pi i \frac{1}{4} \left[e^{-i\omega\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)} \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) + e^{-i\omega\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)} \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \right]$

$\displaystyle = -2\pi i \frac{1}{4} \cdot \frac{1}{\sqrt{2}} \left[e^{\frac{\omega}{\sqrt{2}} - \frac{\omega}{\sqrt{2}} i} (1+i) + e^{\frac{\omega}{\sqrt{2}} + i\frac{\omega}{\sqrt{2}}} (-1+i) \right]$

$\displaystyle = -2\pi i \frac{1}{4} \frac{1}{\sqrt{2}} e^{\frac{\omega}{\sqrt{2}}} \cdot 2i \left(\cos\frac{\omega}{\sqrt{2}} - \sin\frac{\omega}{\sqrt{2}} \right) = \frac{\pi e^{\frac{\omega}{\sqrt{2}}}}{\sqrt{2}} \left(\cos\frac{\omega}{\sqrt{2}} - \sin\frac{\omega}{\sqrt{2}} \right) = \pi e^{\frac{\omega}{\sqrt{2}}} \cos\left(\frac{\omega}{\sqrt{2}} + \frac{\pi}{4} \right)$

• Dus voor $\omega \in \mathbb{R}$ $\quad G(\omega) = \pi e^{-|\omega|/\sqrt{2}} \cos\left(\frac{\pi}{4} - \frac{|\omega|}{\sqrt{2}} \right)$

5. • $n=0$: $\displaystyle 2\pi c_0 = \int_{-\pi/2}^{\pi/2} dt = \pi \Rightarrow c_0 = \frac{1}{2}$; $n \neq 0$: $\displaystyle 2\pi c_n = \int_{-\pi/2}^{\pi/2} e^{-int} dt = 2\int_0^{\pi/2} \cos nt \, dt = \frac{2}{n} \sin n\frac{\pi}{2}$

• Fourier: $\displaystyle \sum_{n=-\infty}^{\infty} c_n e^{int} = \frac{1}{2} + \sum_{n\neq 0} \frac{1}{n\pi} \sin n\frac{\pi}{2} \cdot e^{int} = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi/2}{n} \cos nt$

$\displaystyle = \frac{1}{2} + \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)\frac{\pi}{2}}{2m+1} \cos(2m+1)t = \frac{1}{2} + \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \cos(2m+1)t$

• $t=0$: $\displaystyle 1 = \frac{1}{2} + \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \Rightarrow \frac{\pi}{4} = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}$

• Parseval: $\displaystyle \int_{-1}^{1} |F(t)|^2 dt = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2 = 2\pi \left[\frac{1}{4} + \frac{4}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \right] \Rightarrow \frac{\pi^2}{8} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}$